MATH 303 – Measures and Integration
Homework 4

Problem 1. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to \mathbb{C}$. Suppose

$$\sum_{n\in\mathbb{N}} \int_X |f_n| \, d\mu < \infty.$$

Prove that $\sum_{n\in\mathbb{N}} f_n$ converges a.e. to an integrable function $f \in L^1(\mu)$, and

$$\int_X f \, d\mu = \sum_{n\in\mathbb{N}} \int_X f_n \, d\mu.$$

Solution: By Theorem 3.12,

$$\int_X \sum_{n=1}^{\infty} |f_n| \, d\mu = \sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.$$

Hence, the function $F = \sum_{n=1}^{\infty} |f_n|$ is integrable. In particular, $F < \infty$ a.e. by Proposition 3.20. Therefore, the series $\sum_{n=1}^{\infty} f_n(x)$ converges (absolutely) for a.e. $x \in X$.

Let $S_N = \sum_{n=1}^{N} f_n$. Then S_N is integrable, $S_N \to \sum_{n=1}^{\infty} f_n$ a.e., and $|S_N| \leq F$. Hence, by the dominated convergence theorem, $\sum_{n=1}^{\infty} f_n$ is integrable and

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \lim_{N\to\infty} \int_X \sum_{n=1}^{N} f_n \, d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Problem 2. In this exercise, we will use measure-theoretic tools in order to carry out computations with Riemann integrals. Assume for the purposes of this exercise that there is a measure λ on the Borel subsets of $\mathbb{R}$ with the property: if $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function, then

$$\int_{[a,b]} f \, d\lambda = \int_a^b f(x) \, dx,$$

where the integral on the left is the measure-theoretic integral and the integral on the right is the Riemann integral. (We will discuss multiple methods of constructing such a measure λ (the *Lebesgue measure*) in future lectures.)

(a) Compute

$$\lim_{n\to\infty} \int_0^{\infty} \frac{n \sin\left(\frac{x}{n}\right)}{x(1 + x^2)} \, dx.$$

(b) Show that for $a > -1$,

$$\int_0^1 \frac{x^a \log x}{1 - x} \, dx = -\sum_{k=1}^{\infty} \frac{1}{(a + k)^2}.$$

Solution: (a) Let $f_n(x) = \frac{n\sin\left(\frac{x}{n}\right)}{x(1+x^2)}$. Then for $x > 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin\left(\frac{x}{n}\right)}{\frac{x}{n}} \cdot \frac{1}{1+x^2} = \frac{1}{1+x^2}.$$

Moreover, using the inequality $|\sin t| \leq t$ for $t > 0$, we have a pointwise bound $|f_n(x)| \leq \frac{1}{1+x^2}$. Let $f(x) = \frac{1}{1+x^2}$. By the fundamental theorem of calculus,

$$\int_{[0,\infty)} f \, d\lambda = \int_0^\infty \frac{1}{1+x^2} \, dx = \lim_{a\to 0^+, b\to\infty} (\arctan b - \arctan a) = \frac{\pi}{2} < \infty,$$

so f is Lebesgue integrable. Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^\infty \frac{n\sin\left(\frac{x}{n}\right)}{x(1+x^2)} \, dx = \lim_{n\to\infty} \int_{[0,\infty)} f_n \, d\lambda = \int_{[0,\infty)} \lim_{n\to\infty} f_n \, d\lambda = \int_{[0,\infty)} f \, d\lambda = \frac{\pi}{2}.$$

(b) Expand

$$\frac{1}{1-x} = \sum_{n=0}^\infty x^n$$

for $x \in (0,1)$. Let

$$f_n(x) = x^{a+n} \log x$$

so that

$$\sum_{n=0}^\infty f_n(x) = \frac{x^a \log x}{1-x}.$$

Integrating by parts,

$$\begin{aligned}
\int_0^1 f_n(x) \, dx &= \int_0^1 \log x \; d\left(\frac{x^{a+n+1}}{a+n+1}\right) \\
&= \frac{x^{a+n+1}}{a+n+1} \log x \Big|_0^1 - \int_0^1 \frac{x^{a+n+1}}{a+n+1} \, d(\log x) \\
&= -\frac{1}{a+n+1} \int_0^1 x^{a+n} \, dx \\
&= -\frac{1}{(a+n+1)^2}.
\end{aligned}$$

Hence, noting that $f_n \leq 0$, we have

$$\sum_{n=0}^\infty \int_0^1 |f_n| \, d\lambda = \sum_{n=0}^\infty \frac{1}{(a+n+1)^2} \leq \frac{1}{a+1} + \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

Therefore, applying Problem 1,

$$\int_0^1 \frac{x^a \log x}{1-x}\ dx = \int_0^1 \sum_{n=0}^{\infty} f_n(x)\ dx = \sum_{n=0}^{\infty} \int_0^1 f_n(x)\ dx = \sum_{n=0}^{\infty} \left(-\frac{1}{(a+n+1)^2} \right).$$

Substituting $k = n + 1$ finishes the proof.